Impulse propagation in dissipative and disordered chains with power-law repulsive potentials

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Abstract

We report particle dynamics based studies of impulse propagation in a chain of elastic beads with dissipative contacts and with randomly distributed masses. The interaction between the beads is characterized by the potential \( V(\delta) \sim \delta^n, \delta \geq 0 \) being grain overlap, \( n > 2 \) and at zero external loading, i.e., under conditions of “sonic vacuum” in which sound cannot propagate through the chain [J. Appl. Mech. Technol. Phys. 5 (1983) 733]. In the earlier work, we have confirmed the studies of Nesterenko and coworkers and have reported that impulses propagate as solitary waves in the system of interest in the absence of dissipation and disorder [Physica A 268 (1999) 644]. In the present study, we first discuss the effects of restitution and velocity dependent friction on the propagation of the impulse. We next report that the maximum energy \( E_{\text{max}} \) of the solitary wave as it propagates from a chain of monodisperse grains of mass \( m \) to a chain with masses \( m(1 + r(z)\epsilon) \), where \( -1 \leq r(z) \leq 1 \) and \( \epsilon = \text{const.} \) that measures the degree of randomness, decays with linear distance traveled \( z \) as \( \exp(-\alpha_E z) \), \( \alpha_E \sim \epsilon^{2 + f(n)}, f(n) \) being some \( n \) dependent constant for \( 2 < n < \infty \). In monodisperse chains, the velocity of the solitary wave \( c \sim E_{\text{max}}^{(n-2)/2n} \). In polydisperse chains, we show that the propagation speed of a non-dispersive solitary wave decays with distance as \( \exp(-\alpha_c z) \), where \( \alpha_c = \alpha_E (n-2)/2n \). © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The problem of impulse propagation through a chain of elastic beads has received significant attention since 1983 [1–15]. The most interesting feature of this problem lies in the fact that there is no linear part to the power-law repulsive (PLR) potential that describes the repulsive force between two adjacent grains that are barely in contact. Hence, one cannot invoke the wave equation to study impulse propagation through a chain of uncompressed beads and therefore, the sound velocity through an uncompressed bead chain is zero, as emphasized via the term “sonic vacuum” by Nesterenko [1]. In the presence of loading and hence precompression of the chain, it becomes possible to linearize the repulsive forces between the
elastic beads and hence acoustic propagation becomes possible. Thus, when describing impulse propagation in a chain of elastic beads, the nature of precompression and the initial conditions must be specified. In this work, we consider grains that are not subjected to loading or precompression and an edge bead is subjected to an impulse at some initial time \( t = 0 \).

The elastic beads repel upon contact via the PLR potential, a special case of the PLR potential being the so-called Hertz potential [16],

\[
V(\delta) = \frac{2}{5D} \sqrt{\frac{R}{2}} \delta^n \equiv a \delta^n, \tag{1}
\]

where the beads are of radius \( R \) and \( D \equiv \frac{3}{2}(1 - \sigma^2) / Y \), in which \( \sigma \) and \( Y \) are Poisson’s ratio and Young’s modulus of the beads, respectively. The overlap function \( \delta \equiv 2R - (r_{i+1} - r_i) \geq 0 \). Hertz [16] showed that for spherical beads in contact \( n = \frac{3}{2} \). Subsequent research has demonstrated that for bead contacts with imperfections, typically, \( \frac{5}{2} < n < 3 \) [17]. These special cases of \( n \) are often referred to in the literature as Hertz law. In this paper, we consider a chain of beads in which the nearest neighbor beads interact via the PLR potential (see Eq. (1)), where \( n \) is a parameter that can have continuous values \( 2 < n < \infty \). We assume that the same type of interaction remains valid even in the presence of dissipation. The equation of motion of some bead \( i \) (excluding the edge beads) in a chain is given by

\[
m_i \ddot{u}_i = na[(\Delta - (u_i - u_{i-1}))^{n-1} - (\Delta - (u_{i+1} - u_i))^{n-1}], \tag{2}
\]

where \( m_i \) denotes the mass of bead \( i \), \( u_i(t) \) describes the displacement from equilibrium position and \( \Delta \) is a constant initial loading. We assume that initially the beads barely touch one another. Therefore, \( \Delta = 0 \) at \( t = 0 \) and hence there is \textit{no precompression} in the system. An impulse defined by an initial velocity \( v_1 \neq 0, v_i = 0 \) for \( i > 1 \) at \( t = 0 \) is then initiated. Sufficiently far from the edge, the impulse is assumed to propagate at constant speed \( c \) (this assumption has been verified via numerical and experimental studies). One can hence parameterize distance from the chain edge as \( z = ct, t \) being the elapsed time since \( t = 0 \).

Eq. (2) can then be rewritten as follows:

\[
m(z)c^2 \frac{d^2 \phi_n(z)}{dz^2} = na[(\phi_n(z - d) - \phi_n)^{n-1} - (\phi_n(z) - \phi_n(z + d))^{n-1}], \tag{3}
\]

which is a convenient form for analyzing the dynamical problem. In Eq. (3), \( \phi(z, t) = u_i(t), \phi(z \pm d) = u_{i \pm 1}(t) \), where \( d \) is the distance between the adjacent beads. The subscript \( n \) in \( \phi_n \) is to remind the reader that \( \phi \) depends upon the contact geometry between the beads and hence on \( n \).

Let us assume that \( m(z) = m \). We contend that the following function describes the traveling perturbation and its boundaries accurately:

\[
\phi_n(z) = \frac{A}{2} \left( 1 - \tanh \left( \frac{f_n(z)}{2} \right) \right), \tag{4}
\]

where

\[
f_n(z) = \sum_{q=0}^{\infty} C_{2q+1}(n) z^{2q+1} \tag{5}
\]

and the first three coefficients in \( C_{2q+1}(n) \) can be determined by exploiting the symmetry of the propagating solitary wave and a hybrid analytical–numerical approach consisting of fitting the solution in Eqs. (4) and (5) with the numerically generated solution to Eq. (3) [11]. It is often not necessary to generate more than the first three coefficients. For a given \( n \), the coefficients are material independent. The solution is also better convergent than the only other available solution to Eq. (2), generated in the long wavelength approximation by Nesterenko [1,7]. It turns out that the solution to Eq. (2) describes a traveling solitary wave. From the above analysis, it can be shown [8] that the solitary wave obeys following scaling laws:

\[
c = \sqrt{\frac{na}{m} C_0 \left( \frac{A}{2} \right)^{(n-2)/2}},
\]

\[
v_{\text{max}} = \sqrt{\frac{na}{m} C_0 \left( \frac{A}{2} \right)^{n/2}}, \tag{6}
\]
where \[11\]
\[
C_0(n) = \frac{m c^2}{n a (A/2)^{n-2}}
= [\{\phi_n(z - d) - \phi_n(z)\}^{n-1} - [\phi_n(z) - \phi_n(z + d)]^{n-1}]
\times \frac{d^2 \phi_n(z)}{dz^2}.
\](7)

Since the right-hand side of Eq. (7) is independent of \(m, a\) and solitary wave amplitude \(A\), so is \(C_0\). However, \(C_0\) is a function of \(n\).

Combining the expressions for \(c\) and \(v_{\text{max}}\) in Eq. (6) and eliminating \(A\) yields the following dependence of the solitary wave velocity on the maximum kinetic energy of each grain \(E_{\text{max}}\),
\[
c \sim \frac{E_{\text{max}}^{(n-2)/2n}}{\binom{a}{n}}.
\](8)

These solitary waves possess width \(L(n)\), where \(1 < L(n) < \infty\) for \(2 < n < \infty\) [8]. Hence, it is not appropriate to describe these solitary waves in the long wavelength or continuum approximation. Solitary wave formation is not possible for \(n = 2\), which yields the harmonic chain with a one-sided potential (since \(\delta \geq 0\)). In this paper, we first discuss the effects of restitution and velocity dependent friction on the propagation of the impulse. We next report that the maximum energy \(E_{\text{max}}\) of the solitary wave as it propagates from a chain of monodisperse grains of mass \(m\) to a chain with masses \(m(1 + r(z)\epsilon)\), where \(-1 \leq r(z) \leq 1\) and \(\epsilon = \text{const.}\) that measures the degree of randomness, decays with linear distance traveled \(z\) as \(\exp(-\alpha_E z)\), \(\alpha_E \sim \epsilon^{2+f(n)}\), \(f(n)\) being some \(n\) dependent constant for \(2 < n < \infty\). In monodisperse chains, the velocity of the solitary wave \(c \sim E_{\text{max}}^{(n-2)/2n}\). In polydisperse chains, we show that the propagation speed of a non-dispersive solitary wave decays with distance as \(\exp(-\alpha_c z)\), where \(\alpha_c = \alpha_E(n - 2)/2n\). We show, via analytical arguments and dynamical simulations that the solitary waves in chains with PLR are robust objects.

2. Impulse propagation in dissipative chains

The system under investigation consists of 2000 grains. The first 1000 of these are monodisperse (with \(a = 1\) and \(m = 1\)). The next 1000 grains possess dissipative contacts or certain types of disorder. A perfect solitary wave with maximum kinetic energy (= 1) is produced in the first (i.e., pristine region) half of the system. The maximum kinetic energy is obtained by recording the maximum kinetic energy of one grain as a function of time which is proportional to the total kinetic energy and the total energy of the system.

In this section, we study the impulse propagation in a chain made out of dissipative elastic grains. The inelasticity is introduced via a restitution coefficient [18], which is defined by
\[
\frac{F_{\text{unloading}}}{F_{\text{loading}}} = 1 - w.
\](9)

Thus, the contact gets compressed with the force \(F_{\text{loading}}\), but decompresses with less force at unloading. Hence, a part of the energy remains trapped in the internal modes of the grain. We observe that \(w = 0\) corresponds to the perfectly elastic case.

In Fig. 1, the kinetic energy is plotted as a function of time for different particles for \(w = 0.02\). The incident solitary wave is recorded at particle 950 (the grain contacts with restitutional losses start at contact 1000). It can be seen that there is a dramatic decrease in the magnitude of the solitary wave due to restitutional energy loss. While such attenuation in the amplitude of the signal is expected, it is important to observe that the impulse propagates without dispersion. It turns out that the pulse at each grain is well described by the same solitary wave function (Eqs. (4) and (5)) with the progressively decaying amplitude of the solitary wave depending on its position in the chain. Hence, the propagation of an impulse in a dissipative chain of beads that interact via the PLR potential is completely described if one obtains an equation to compute amplitude as a function of distance.

In Fig. 2, for various restitution coefficients \(w\), we compute the maximum kinetic energy \(E(z)\) for each grain located at a distance \(z \equiv 2NR\), where \(N\) denotes the distance (in grain diameters) from the interface between monodisperse and polydisperse chains with \(N = 1\) being the first grain in the polydisperse chain. It can be seen that the maximum kinetic energy of the pulse as it travels through the polydisperse chain,
$E(z)$, is well described by an exponential decay in $z$. We find that at least across three decades with the decay coefficient depending on $w$,

$$E(z) = E_0 \exp(-\alpha_E(w)z),$$

(10)

where $E_0$ represents the maximum kinetic energy of the solitary wave before it enters the polydisperse region. The subscript $E$ indicates that the coefficient $\alpha_E$ stands for energy decay. Eq. (10) coupled with the scaling laws for solitary waves (Eqs. (6)–(8)) implies that the amplitude, propagation velocity and maximum velocity obey the same exponential decay law with appropriate coefficients.

In order to test the dependence of $\alpha_E$ on $w$, we repeated the same analysis for systems with different $n$’s. Such an analysis yields solitary waves with different widths and hence affects the nature of restitutional losses at contacts. The results are

Fig. 1. Kinetic energy vs. time for different beads. The solitary wave decreases (because of energy loss) but no background noise is induced.

Fig. 2. Ratio between maximum bead energy and maximum incident energy vs. bead number for different restitution coefficient $w$. 
Fig. 3. Energy decay constant ($\alpha_E$) for different restitution coefficients $w$ and interaction power law constant $n$. The data reveal that for sufficiently small values of $n$, the analytic prediction fails.

The results of Fig. 3 suggest how the problem may be analytically tackled. Let us suppose that one has an ideal chain with only one dissipative interface. As shown before, these interfaces do not reflect solitary waves, but only decreases the amplitude of the incident solitary wave by trapping a part of the energy via the restitutional loss mechanism (we assume that $w$ is small enough). The kinetic energy of the transmitted solitary wave is then proportional to the kinetic energy of the incident one, $E(z) = E_0(1 - \xi)$, where $\xi$ represents the fractional energy loss.

Suppose now that we have multiple dissipative contacts, say, every $q$ grains, where $q > L(n)$. In this case, all the collisions would be independent, and the energy after $N$ interfaces will be given as follows:

\[ E(z) = E_0(1 - \xi)^z \]
\[ = E_0 \exp(\ln(1 - \xi)z) \approx E_0 \exp(-\xi z), \]  

the last approximation being valid for $\xi \ll 1$.

In the case of successive dissipative interfaces, the situation is more complex because the solitary wave interacts with more than one interface at the same time. However, for large enough $n$, it can be assumed that most of the elastic energy of the solitary wave encounters only one contact at a given time which implies that Eqs. (10) and (11) remain approximately valid.

The next question to address is, what is the relation between energy loss at a contact $\xi$ and restitution coefficient $w$? In a system where one grain is fixed, and another grain compresses into it and then recoils, the contact gets loaded and then moves back. The fractional energy loss is of course $\xi = w$. This is no longer true in the dynamic chain, where the grains travel during force application, and hence the total work done on an individual grain by its neighbor is different from $\xi$. To clarify this notion, in Fig. 4, we represent the force at the contact and the displacements $u_1, u_2$ of two adjacent beads as function of time. The mechanical work transmitted from one particle to another is $\int F(u_2) du_2$, which is the area under the curve presented in Fig. 5. The continuous line represents the
Fig. 4. Displacement of adjacent grains $u_1(t), u_2(t)$ and the PLR force between them as a function of time for a solitary wave. All the units are arbitrary.

Fig. 5. Inter-grain force vs. second bead position. It is assumed that the force is transmitted instantaneously through the bead (speed of sound in bead is much greater than speed of solitary wave propagation).
force in an ideal contact with PLR \((w = 0)\), while the circles, squares and triangles represent the corresponding force for \(w = 0.01, 0.05\) and \(0.10\), respectively. Neglecting the small deformation of the force curve in non-ideal cases, the mechanical work transmitted by the interface can be computed using the known function for displacement and force. It is considered that \(E(0)\) is transmitted in an ideal chain with PLR interaction between the grains (the area under continuous line in Fig. 5), while \(E(0)(1 - \xi)\) is transmitted by a dissipative contact (the area under the other lines). The result for \(n = \frac{5}{2}\), using our asymptotic solution from Eq. (4), is \(\xi = 0.38w\). The linearity in \(w\) is a consequence of neglecting the shape modification of the force. From Fig. 5, it is seen that this approximation is reasonable for small \(w\). The approximation that almost all of the potential energy is localized in only one contact at a given time is clearly better for higher \(n\), since the widths of the solitary waves shrink to 1 grain diameter as \(n\) diverges. The approximation is expected to be less accurate for \(n \rightarrow 2\), in which case, the size of the solitary wave diverges. It can be seen from Fig. 3 that the approximation is already poor for \(n = 2.2\).

For \(n = \frac{5}{2}\), the agreement between the analytical result (continuous line in Fig. 3) and numerically computed results are excellent, at least for \(w \leq 0.1\). This will imply in turn that for a given system, in which the material parameters are known \((\alpha, m, n \text{ and } w)\), one can analytically compute, using Eqs. (4), (5) and (11), the position, velocity and shape of a solitary wave at any point in space and time.

In this section, we have studied the behavior of the impulse in a chain with PLR interactions upon compression and with dissipative particles, where the loss of energy occurs via an incomplete restitution mechanism. A similar analysis can be made when the energy loss occurs via external mechanisms such as friction. In Fig. 6, we present the maximum kinetic energy as a function of grain position in the presence of velocity dependent friction \(f = \gamma v\). For small values of \(\gamma\), the decay of energy is well approximated, again, by an exponential function. The decay coefficient \(\alpha_E\) is found to depend linearly on the friction coefficient.
Fig. 7. Energy decay constant $\alpha_{E}$ for different friction coefficients ($n = \frac{5}{4}$). The dependence is linear.

across the whole region investigated as shown in Fig. 7.

3. Impulse propagation in disordered chains

In this section, we study the problem of impulse propagation in polydisperse chains. As before, the system consists of 2000 grains, with the first 1000 being perfectly elastic and monodisperse. The other 1000 grains have randomly distributed values of density, size, elastic constant or $n$. Our extensive numerical investigations show that all the different types of randomness lead to qualitatively similar results. Since randomness in size distribution implies randomness in mass and randomness in the interaction constant $a$ for each grain, the most drastic effect is produced by randomness in size distribution. Here, we address solitary wave dynamics in a chain where the grain masses can be described as a function of position $z$ by $m(z) = m_0 (1 + r(z) \epsilon)$, where $r$ is a random number uniformly distributed between $-1$ and 1, and $\epsilon$ is a fixed parameter between 0 and 1 which describes the degree of randomness of the system.

A perfect solitary wave is generated in the monodisperse half of the chain and propagates towards the random medium. A typical result from a particle dynamics simulation with $\epsilon = 0.15$ is shown in Fig. 8. In Fig. 8(a) and (b), the kinetic energy is plotted for each grain at two different times: (a) before the impulse enters the random region; (b) after it propagates into the random region. While Fig. 8(a) deserves little explanation, in Fig. 8(b) it is observed that part of the energy remains behind the leading edge of the signal as noise. When the noise hits the monodisperse region, it is quickly decomposed into solitary waves, as expected [8,9]. The most interesting feature of impulse propagation is that the leading edge maintains its solitary-wave-like shape and is non-dispersive. This result is also supported by the numerical analysis of Nesterenko [1]. We have found that this property remains valid even after the amplitude of the leading edge decays by orders of magnitude. The reason for
this behavior lies in the fact that the width (measured in grain diameters) of the solitary wave propagating in an ideal chain is independent of the grain parameters and the solitary wave amplitude [11]. The width depends upon the index of the PLR potential \(n\); however, the variations are small for \(n\) far from the harmonic limit \(n = 2\) [8,9]. We focus upon the effects of polydispersity on the trailing portion of the propagating wave below.

While the solitary wave is about three grain diameters wide (for \(n = \frac{5}{2}\)), at a given point in time most of the energy is carried by only one grain. Let us think of solitary waves as being carried entirely by only one grain at a time. When the solitary wave encounters the first grain with different density, a part of the energy will be transmitted and a part will be reflected [9,10]. The process will be repeated at successive granular contacts. Hence, the leading edge will continue propagation and will be attenuated at each contact.

The reflected impulses also suffer backscattering and these multiple reflection/transmission processes lead to the noise behind the leading edge. Because the reflected impulses are much smaller than the transmitted ones (for reasonable levels of randomness), the multiple overlapping impulses, which constitute the noise, propagate with less velocity than the leading edge (recall that the velocity depends on amplitude.

Fig. 8. Maximum kinetic energy of each bead of the chain (a) before and (b) after entering the disordered region. Kinetic energy as a function of time for (c) bead 950 and (d) bead 1100.
via Eq. (6)). Hence, the propagating leading edge does not interact with the noise it produces, and the pulse decays because of loss of energy, without suffering any shape alterations. Of course, this rough picture becomes less accurate for smaller $n$ because of the increase of the width of the solitary wave.

In order to test that the leading edge has the shape of the solitary wave and that the noise behind the leading edge can be thought as being made by multiple reflected solitary waves, in Fig. 8, we plot the kinetic energy as a function of time for grains 950 (Fig. 8(c)) and 1100 (Fig. 8(d)). The analysis suggests that the leading edge of an impulse will propagate as a solitary-wave-like object, whose amplitude will exhibit an approximate exponential decay for relatively low randomness.

We have run the simulations described above for various values of randomness and we have recorded the maximum kinetic energy for each particle in turn. The results of such a computation are presented in Fig. 9. It can be seen that even for high degrees of disorder, the average exponential decay behavior is respected.

The maximum kinetic energy of each grain depends also on the mass of that grain, which has random values. This is the reason why the points representing maximum kinetic energy are spread around the average exponential decay. One can also measure the deviation $\sigma_E$ of these points from the average exponential decay of the energy characterized by $E(z) = E_0 \exp(-\alpha_E z)$ by using the data in Fig. 9. Our extensive simulations indicate that the decay coefficient $\alpha_E$ and the statistical dispersion $\sigma_E$ depend on the randomness of the chain $\epsilon$, but are independent of the region of measurement. This implies in turn that the pulse propagation remains qualitatively unchanged over the whole interval studied, and again can be described by simple measurable parameters.

In Fig. 10, we present the values of $\alpha_E$ and $\sigma_E$ obtained from our simulational data for different randomness and for $n$ ranging from 2.2 to 5.0. While the statistical dispersion $\sigma_E$ is (approximately) a linear
function of \( \epsilon \), the exponential coefficient of decay in energy, \( \alpha_E \) in \( E(z) \approx E_0 \exp(-\alpha Ez) \), has a power law dependence on randomness \( \epsilon \) with a coefficient \( \approx 2 \), i.e., \( \alpha_E \propto \epsilon^{2+f(n)} \). The exponential relationship implies that for small enough \( \epsilon \), \( E(z) = E_0(1 - \epsilon^2)^z \) (see Eq. (11)). Our calculation indicates that the power exponent is close to 2 for all the cases investigated, which is justified by the analysis of the energy transmitted at the interface between two ideal chains with PLR potential at grain–grain interfaces (see Appendix A).

An interesting question is, how the maximum energy is distributed among the random grains? Of course, we can ask this question only in a statistical sense, since the kinetic energy that one grain receives depends on the incoming energy and on the random values of the mass of its neighbors. In Fig. 11(a), we represent the maximum kinetic energy vs. the mass of each particle for a pulse propagating in a chain with \( n = 5 \frac{1}{2} \) and \( \epsilon = 0.15 \). It can be inferred that the particle with higher mass acquires more kinetic energy than the particle with smaller mass. This result seems to be paradoxical (since higher mass particle should be moving slower), but Fig. 11(b) shows that, indeed, the higher mass particles reach lower maximum velocity than their less massive counterparts.

The study of dependence of maximum kinetic energy on the random particle mass brings additional insight into one-particle approximation of
the pulse propagation. If the solitary wave is one grain wide, the energy will flow from one grain to the other. In this case, there will be no statistical dispersion of energy, but only energy loss, as predicted by the law of conservation of energy and momentum for collision of elastic objects with different mass. On the other hand, if the solitary wave is very broad, there are many body effects and the impulse has the possibility to adjust itself such as to distribute more kinetic energy to the heavy particles. This hypothesis is confirmed by the studies depicted in Fig. 11 for $n = 2.2$ (Fig. 11(c)) and $n = 5.0$ (Fig. 11(d)).

In conclusion, in random media the excitation splits into two parts: a solitary-wave-like leading pulse, which propagates without dispersion, but with its amplitude exponentially decreasing because of energy loss via multiple backscattering processes, and a noise which lags the leading pulse. The dynamics of the leading edge of the propagating pulse is described by two additional parameters, the coefficient of average exponential attenuation $\alpha_E$ and the statistical dispersion $\sigma_E$. As an application, we will show how the knowledge of these parameters will allow one to probe the pulse propagation. Assuming exponential decay of maximum kinetic
energy given by Eq. (10) and using the scaling laws (Eqs. (6)–(8)), the velocity attenuation of the pulse is given by

\[ c = c_0 \exp(-\alpha_c z), \quad \alpha_c = \frac{n - \frac{2}{2n} \alpha_E}{}, \] (12)

where \( c_0 \) refers to the velocity of the solitary wave before it entered the polydisperse segment of the chain.

The position \( z \) of the pulse after entering the random region can be obtained by integrating Eq. (12):

\[ z = \frac{1}{\alpha_c} \ln(\alpha_c c_0 t + 1). \] (13)

In Fig. 12, the position of the pulse is computed for chains with different degrees of randomness, and is compared with Eq. (13). The coefficient \( \alpha_E \) is computed separately, which means that there is no parameter optimization in Eq. (13). In all the cases, the agreement is good, which implies that the two additional parameters (\( \alpha_E \) and \( \sigma_E \)) offer a good description for the propagation of solitary waves in random media.

4. Impulse propagation in dissipative random chains

In this section, we investigate impulse propagation in disordered chains made out of dissipative beads. The dynamics of the dissipative grains is simulated via the restitution coefficient, as described in Section 2. The disorder is modeled as in Section 3 by attributing a random value to the mass of each bead. Again, our calculations indicate that the leading pulse maintains its width but its amplitude is exponentially attenuated. The results of the calculations, for \( n = \frac{5}{2} \) and different degrees of randomness \( \epsilon \) and restitution \( w \) are presented in Fig. 13. The coefficient of attenuation \( \alpha_E \) increases as a result of both types of disorder, while the statistical dispersion \( \sigma_E \) is not affected by the restitution coefficient.

We hope that these studies would stimulate new experiments that would validate our calculations. Such confirmation might lead one to design materials with desired signal transmission properties and hence with
Fig. 13. (a) Dispersion $\sigma_E$ and (b) decay constant $\alpha_E$ for a chain with $n = 2.5$ and different randomness $\epsilon$ vs. restitution coefficient $w$. The statistical dispersion is a measure of randomness, while the decay constant depends on both randomness and restitution.

wide ranging applications in materials science and engineering.

5. Conclusions

This paper presents studies of impulse propagation in dissipative and disordered chains with PLR potential describing the grain–grain interactions. It is well known that ideal chains, with identical perfect elastic beads, support solitary waves [1,11]. Here we show that even in the presence of disorder, the impulse travels as a compact object. The leading edge of the signal maintains its original width, while its amplitude decreases exponentially with distance. This decay is due either to the energy loss due to restitution or friction processes, or to multiple backscattering of the signal in the presence of randomness in bead parameters. The coefficient of exponential decay is quantitatively related to the degree of imperfection (either restitution $w$, friction coefficient $\gamma$ or randomness $\epsilon$).

In the presence of randomness in the bead parameters, an important quantity is the statistical dispersion of the maximum kinetic energy around the average kinetic energy. It is shown that these two parameters (in addition to the equations of the solitary waves of the monodisperse chains) offer a good description of the problem of impulse propagation. The different dependencies of these parameters on the particular mechanism of energy loss suggest that our studies
may be useful in characterizing disordered chains with PLR interactions. Notwithstanding the calculational difficulties that must be surmounted, extension of this work to 2D and 3D systems is hence of interest and is being currently pursued [19].

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Appendix A. Arguments to establish \( \alpha(\epsilon) \propto \epsilon^2 \)

We consider a chain of grains with randomly distributed masses \( m(\epsilon) = (1 + r(z)\epsilon)m \), where \(-1 \ll r(z) \ll 1\) and \( r \) is a random number and \( \epsilon \) is a constant that controls the degree of randomness. We start by observing that the velocity of a solitary wave \( v_{\text{max}} \) of the fastest moving grain at any instant in a solitary wave is given by Eq. (6) and this implies that the kinetic energy associated with the solitary wave is approximately

\[
\text{KE} \approx \frac{1}{2}m \left[ \frac{na}{m} C_0(n) \frac{C_1}{2} \left( \frac{A}{2} \right)^{n/2} \right]^2 \propto A^n. \quad (A.1)
\]

The potential energy also has the same dependence on the amplitude. However, we need not consider the potential energy for our purposes. The momentum is given by

\[
p \approx mv_{\text{max}} \propto m \sqrt{\frac{na}{m} C_0(n) \frac{C_1}{2} \left( \frac{A}{2} \right)^{n/2} } \propto \sqrt{mA^n/2}. \quad (A.2)
\]

In the above analysis, we assume that the solitary wave is 1 grain wide instead of 3 grains wide. Since most of the energy is carried by a single grain, this is a reasonable approximation to achieve the level of understanding that we are seeking. We now consider the motion of an energy bundle through the chain of randomly distributed masses.

We now impose conservation of energy and of linear momentum and argue that if \( A' \) and \( A'' \) denote the transmitted amplitude and the backscattered amplitudes, one can show that

\[
A'' = (A')^n + (A'')^n, \quad (A.3)
\]

and

\[
A'^{n/2} = \sqrt{1 + \epsilon (A')^{n/2} - (A'')^{n/2}}, \quad (A.4)
\]

and hence one finds

\[
A' = A \left( \frac{2 \sqrt{1 + \epsilon}}{2 + \epsilon} \right)^{2/n}, \quad (A.5)
\]

and

\[
A'' = A \left( \frac{\epsilon}{2 + \epsilon} \right)^{2/n}. \quad (A.6)
\]

When one considers small values of \( \epsilon \), it turns out that

\[
\text{KE'} \propto A'^n \propto A^n \left( 1 - \frac{1}{4} \epsilon^2 \right) \propto \text{KE}(1 - \alpha_E), \quad (A.7)
\]

and hence

\[
\alpha_E = \epsilon^2. \quad (A.8)
\]

References